An intermediate case of exponential multivalued forbidden matrix configuration

Wallace Peaslee[a], Attila Sali[b], Jun Yan[c]

[a]University of Cambridge, Department of Applied Mathematics and Theoretical Physics, wallace.peaslee@gmail.com
[b]HUN-REN Alfréd Rényi Institute of Mathematics, sali.attila@renyi.hu
[c]University of Warwick, Mathematics Institute, j228yan@uwaterloo.ca

The *forbidden number* $\text{forb}(m, F)$, which denotes the maximum number of unique columns in an m-rowed $(0,1)$-matrix with no submatrix that is a row and column permutation of F, has been widely studied in extremal set theory. Recently, this function was extended to r-matrices, whose entries lie in $\{0, 1, \ldots, r-1\}$. $\text{forb}(m, r, F)$ is the maximum number of distinct columns of an r-matrix with no submatrix that is a row and column permutation of F. While $\text{forb}(m, F)$ is polynomial in m, $\text{forb}(m, r, F)$ is exponential for $r \geq 3$. Recently, $\text{forb}(m, r, F)$ was studied for some small $(0,1)$-matrices F, and exact values were determined in some cases. In this paper we study $\text{forb}(m, r, M)$ for $M = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is the smallest matrix for which this forbidden number is unknown. Interestingly, it turns out that this problem is closely linked with the following optimisation problem. For each triangle in the complete graph K_m, pick one of its edges. Let m_e denote the number of times edge e is picked. For each $\alpha \in \mathbb{R}$, what is $H(m, \alpha) = \max \sum_{e \in E(K_m)} \alpha^{m_e}$? We establish a relationship between $\text{forb}(m, r, M)$ and $H(m, (r-1)/(r-2))$, and in the case $r = 3$, prove lower and upper bounds for $H(m, 2)$ and use it to bound $\text{forb}(m, 3, M)$ away from known general upper and lower bounds.